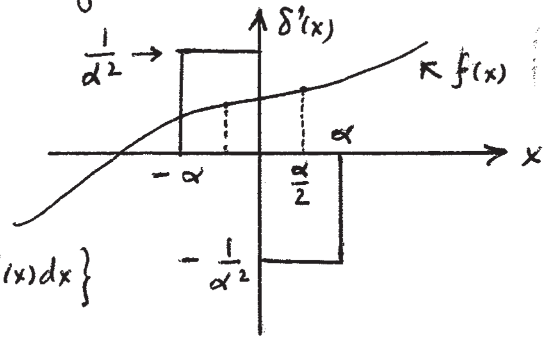
Problem 4) Let $\delta(x) = \lim\limits_{\alpha\to 0} \frac{1}{\alpha}\mathrm{Tri}\left(\frac{x}{\alpha}\right) = \lim\limits_{\alpha\to 0} \frac{1}{\alpha}\left(1 - \frac{|x|}{\alpha}\right)$ when $|x| < \alpha$,

and zero otherwise. Differentiating both sides of the above equation yields:

$$\delta'(x) = \lim_{\alpha\to 0} \begin{cases} \frac{1}{\alpha^2} & -\alpha < x < 0 \\ -\frac{1}{\alpha^2} & 0 < x < +\alpha \\ 0 & \text{otherwise.} \end{cases}$$



$$\int_{-\infty}^{\infty} f(x)\,\delta'(x)\,dx = \lim_{\alpha\to 0}\left\{\int_{-\alpha}^{0} \frac{1}{\alpha^2} f(x)\,dx - \int_{0}^{\alpha} \frac{1}{\alpha^2} f(x)\,dx\right\}$$

$$\simeq \lim_{\alpha\to 0}\left\{\frac{1}{\alpha^2} f\left(-\frac{\alpha}{2}\right)\alpha - \frac{1}{\alpha^2} f\left(+\frac{\alpha}{2}\right)\alpha\right\}$$

$$= \lim_{\alpha\to 0} \frac{f(-\alpha/2) - f(\alpha/2)}{\alpha} = -\lim_{\alpha\to 0} \frac{f(\alpha/2) - f(-\alpha/2)}{\alpha} = -f'(x=0).$$

✓ Another proof using the method of integration by parts:

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x)\,dx = f(x)\,\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx$$

Noting that $\delta(x) = 0$ at $x = \pm\infty$, and using the sifting property of $\delta(x)$ we'll find: $\int_{-\infty}^{\infty} f(x)\,\delta'(x)\,dx = -f'(0)$.

✓ A third proof using Fourier Transforms and the Convolution theorem:

$$\delta(x) = \int_{-\infty}^{\infty} e^{+i2\pi s x}\,ds \Rightarrow \delta'(x) = \int_{-\infty}^{\infty} i2\pi s\, e^{i2\pi s x}\,dx \Rightarrow \mathcal{F}\{\delta'(x)\} = i2\pi s.$$

Let $g(x) = f(x) * \delta'(x)$. Then $G(s) = F(s)\,\mathcal{F}\{\delta'(x)\} = i2\pi s\,F(s) = \mathcal{F}\{f'(x)\}$.

$$\Rightarrow f(x) * \delta'(x) = f'(x) \Rightarrow \int_{-\infty}^{\infty} f(x')\,\delta'(x - x')\,dx' = f'(x) \Rightarrow \int_{-\infty}^{\infty} f(x')\,\delta'(-x')\,dx' = f'(0)$$

Since $\delta'(\cdot)$ is an odd function of x', we have $\delta'(-x') = -\delta'(x')$. The proof is then complete.